ZDM – Mathematics Education (2024) 56:965–980
https://doi.org/10.1007/s11858-024-01597-5

ORIGINAL PAPER

High-quality use of representations in the mathematics classroom – a matter of the cultural perspective?

Anika Dreher[1] · Ting-Ying Wang[2] · Paul Feltes[1] · Feng-Jui Hsieh[2] · Anke Lindmeier[3]

Accepted: 25 May 2024 / Published online: 10 June 2024

Abstract
The teacher's use of representations is a crucial aspect of instructional quality in mathematics education, given their pivotal role in facilitating mathematics learning. However, in our international research community, perspectives on what constitutes high-quality use of representations may vary. This cross-cultural study aims to explore whether the perspectives from Western literature, emphasizing the importance of explicit connections between symbolic and graphic representations, can be extended legitimately to the East Asian context. Using a situated approach, the study elicited norms of high-quality representation use from researchers in Germany and Taiwan. A total of 31 mathematics education professors from both countries evaluated the use of representations in three secondary mathematics classroom situations presented as text vignettes. The vignettes, designed by the German research team, each depicted a situation where from their perspective, a norm of high-quality representation use, specifically the explicit connection between symbolic and graphic representations, was violated. Qualitative analysis of the researchers' responses revealed that in each situation, at least half of the German researchers expected explicit connections between representations. Conversely, the majority of Taiwanese researchers only expected such connections in one situation, particularly when the graphic representation served as an independent learning objective rather than solely aiding conceptual understanding. These findings indicate easily unnoticed culture-specific differences regarding how a common aspect of instructional quality is expected to unfold in teaching.

Keywords Instructional quality · Cultural norms · Representations · Researchers · Intercultural validity · Cross-cultural research

1 Introduction

In some Western frameworks for instructional quality in mathematics classrooms, making connections between representations is emphasized as an important aspect of the teachers' use of representations (e.g., Hill et al., 2008; Schoenfeld, 2018). This aligns with scholars and national standards that highlight recognizing connections between representations as a significant overarching learning goal in mathematics (e.g., Boaler, 2002; KMK, 2003; NCTM, 2000). Empirical studies also indicate that multiple representations are particularly beneficial for students' learning if students are explicitly guided to make connections between them (e.g., Rau & Matthews, 2017; Renkl et al., 2013). However, other scholars in mathematics education have focused on selecting appropriate representations for students' learning when examining teachers' use of representations (e.g., Kunter et al., 2013). Thus, it remains unclear whether making explicit connections can indeed be considered a norm of *high-quality use of representations* in the mathematics education community in the sense that scholars expect teachers to prioritize such connections in specific classroom situations. Moreover, in East Asian countries like Taiwan, neither the national standards nor mathematics education literature typically emphasize the importance of making explicit connections between representations (e.g., Hsieh, 2013; National Academy for Educational Research, 2018). It is thus an open question whether the Western research advocacy of making explicit connections between representations in the mathematics classroom transfers to the East Asian context.

✉ Anika Dreher
anika.dreher@ph-freiburg.de

1 Freiburg University of Education, Freiburg, Germany

2 National Taiwan Normal University, Taipei, Taiwan

3 Friedrich-Schiller-University Jena, Jena, Germany

Therefore, this study seeks to investigate the expectations of German and Taiwanese researchers concerning teachers' use of representations in the mathematics classroom, specifically examining whether making explicit connections between representations can be considered a norm of instructional quality in the mathematics education communities of both countries. To achieve this aim, we use a situated approach, focusing on mathematics education researchers' evaluations of specific classroom situations presented in vignettes.

2 Theoretical background

2.1 Perspectives on instructional quality in Western vs. East Asian countries

Instructional quality is a central construct in mathematics education research, but its conceptions are partly normative and vary between cultural contexts (e.g., Leung, 2001; Liu et al., 2023; Xu & Clarke, 2019), as teaching is a cultural activity (e.g., Coleman, 1990; Hofstede, 1986). For instance, Cortazzi and Jin (1996) described how Western scholars criticized East Asian learners for being passive and for their rote learning due to a lack of understanding of the cultural context. In fact, in East Asian cultures, silence (not speaking) is considered to promote thinking and memorizing is not seen as a contradiction to reflecting and understanding (Cortazzi & Jin, 1996; Xu & Clarke, 2019). In this light, Xu and Clarke (2019) pointed out that the Western research advocacy of engaging students in mathematics classroom dialogue cannot simply be extended to other cultural contexts and is not applicable to the East Asian context. With an analysis of classroom discourse in Western and East Asian classrooms from the Learner's Perspective Study they demonstrated interesting variations in the extent to which discursive practices afforded or constrained the opportunities for students to speak mathematically and its consequences for their learning.

These examples illustrate a problem in our international mathematics education community that Clarke (2013) raised: Research instruments are often shaped by the researchers' culture-specific norms, as are the analyses and conclusions. Moreover, research is often designed from a "Western" perspective using "Western" criteria of instructional quality, which may not be applicable to the East Asian context. A related phenomenon that may be even more prevalent is that aspects of instructional quality are interpreted differently depending on the cultural context: For instance, Mok (2006) analyzed a Chinese mathematics classroom to point out that it may be conceived as student-centered if student-centeredness was understood in a culture-specific way. To figure out how instructional quality is conceptualized in the international mathematics education community and how norms of instructional quality vary between cultural contexts, we hence need to focus on specific classroom interactions.

Following Herbst and Chazan (2011), we use the notion of a *norm* "in the sociological sense as the normal or unmarked behavior that is tacitly expected in a setting" (p. 411). In particular, *norms of instructional quality* are "norms of interaction between teacher, students, and specific content" (p. 424). As education is part of an overarching cultural context, norms of instructional quality are part of a wider cultural norm system (Coleman, 1990). With *culture*, we refer to the "shared motives, values, beliefs, identities, and interpretations or meanings of significant events that result from common experiences of members of collectives that are transmitted across generations" (House et al., 2004).

Accordingly, in his search for an East Asian identity in mathematics education, Leung (2001) identified common features of mathematics education in East Asian countries that are based on a "deep-rooted common set of values" (p. 36). He used the following six dichotomies to contrast characteristics of East Asian and Western mathematics education: product vs. process; rote learning vs. meaningful learning; studying hard vs. pleasurable learning; extrinsic vs. intrinsic motivations; whole class teaching vs. individualized learning; and competence of teachers: subject matter vs. pedagogy. Particularly, the first two dichotomies point to different perspectives on learning goals in mathematics classrooms: While Western scholars emphasize the process of doing mathematics as more relevant than the content arising out of the process, East Asian scholars rather believe that ultimately the content and its correctness are fundamental. Furthermore, whereas the dominant Western view of constructivist learning considers memorization without understanding not as meaningful learning, in East Asia, memorization plays an important role, particularly in view of the exam-driven culture (Leung, 2001). The last two dichotomies point to different understandings of the role of the teacher: While in Western countries, the teacher is often seen as a facilitator focusing on the needs of the individual student, in East Asian countries, the teacher is rather seen as a scholar serving as a role model for the whole class (Leung, 2001). In line with these interpretations, based on a literature review, Kaiser and Blömeke (2013) pointed out differences in the cultural perspectives on teacher expertise: While Western perspectives often focus on interactions with individual students, East Asian perspectives focus more strongly on the teacher as an expert for the mathematics content.

Such differences are not only indicated by research in mathematics education but also supported by theories in cultural psychology, such as the seminal work by Hofstede (1986). He applied his cultural dimensions theory to the context of education and concluded that differences between countries regarding the identified cultural dimensions imply that certain differences in teaching and learning are to be expected. The collectivism-individualism dimension, which appears particularly relevant for explaining differences between East Asian and Western countries, is expected to predict, for instance, not only whether individual students will speak up in class, but also what students expect to learn and whether acquiring competence or passing exams is more important (Hofstede, 1986). From this, Kaiser and Blömeke (2013) concluded that even different perspectives on the nature of mathematics and corresponding learning goals (product vs. process) in East Asian and Western countries may be explained by the collectivism-individualism dimension.

While such general differences may impact expectations of teachers' behavior, these abstract and overstated dichotomies cannot straightforwardly predict what is expected from teachers regarding specific aspects of instructional quality in Western and East Asian mathematics classrooms: For instance, we found in Dreher et al. (2021) that German and Taiwanese researchers' evaluations of how a teacher responded to students' thinking in a classroom situation revealed that regardless of their cultural background, the professors expected the teacher to attend to the individual student's thinking. Nevertheless, different purposes for attending to this student's thinking were identified: Most of the Taiwanese researchers assumed that the student's answer indicated an inappropriate strategy, which should be addressed, whereas most of their German counterparts assumed that the student's answer showed a problem-solving strategy, which should be valued. As reasoned in Dreher et al. (2021), these different purposes and corresponding expectations of how the teacher should respond to this student's thinking could be interpreted in the light of Leung's (2001) dichotomies regarding different learning goals in mathematics education.

Against this background, there is a need for research to uncover how understandings of aspects of instructional quality may vary systematically between cultural contexts in our international mathematics education community.

2.2 The use of representations as an important aspect of instructional quality of mathematics classrooms

The teachers' use of representations in the mathematics classroom is considered an important aspect of instructional quality not only in Western but also in East Asian literature (e.g., Hsieh, 2013; Huang & Cai, 2011; Leong et al., 2015). Representations serve significant functions in learning mathematics, as briefly outlined below. Subsequently, we discuss the teacher's use of representations and potential cultural differences in the relevance of making connections between them.

2.2.1 Functions of representations for learning mathematics

Doing mathematics requires using representations, as mathematical concepts are not accessible without them (e.g., Duval, 2006). These *representations* stand for the abstract mathematical concept and make visible different aspects of it (e.g., Goldin & Shteingold, 2001). There are multiple representations for mathematical concepts with different characteristics that have been categorized in many ways (Ainsworth, 2006). Interculturally well-known categorizations refer to the mode or the level of abstraction of representations: Peirce (1906) initially distinguished between symbolic (language-based) and iconic (graphic-based) representations, with Bruner (1966) adding an enactive (action-based) mode. Bruner's categorization is widely used and was, for instance, the basis for the Concrete-Pictorial-Abstract (CPA) approach advocated in Singapore since the early 1980s (Leong et al., 2015). Bruner's modes can be understood as representing different levels of abstraction, although abstraction may also vary within one mode, for instance, with rather concrete or abstract iconic representations (Purchase, 1998). While there is variability in the grain size of the dimension concrete vs. abstract, the significance of this dimension is widely recognized (Ainsworth, 2006). Accordingly, also the model for mathematics teaching competence proposed by Hsieh (2013) in her search for an East Asian identity in pedagogical content knowledge distinguishes explicitly "concrete versus abstract math representations" (p. 936) regarding the teacher's use of representations.

As in secondary mathematics classrooms, enactive representations often play a minor role, we focus on the other two modes and use the notions of symbolic and graphic representations. A *symbolic representation* is language-based (every-day or formal mathematical language) and abstract in the sense that it usually has no perceptual similarity to what it represents, which facilitates mathematical generalization (Bruner, 1966). A *graphic representation* of a mathematical concept is a spatial configuration that can stand for this concept due to common structural features (Edens & Potter, 2008; Schnotz & Bannert, 2003). These structural features of the graphic representation allow to read off relational information. Thus, graphic representations can often represent the structure of a mathematical concept in an intuitively accessible way (Arcavi, 2003).

In her framework for learning with multiple representations, Ainsworth (2006) delineated *functions of multiple representations* in supporting learning that apply to the use of symbolic and graphic representations: complementing, constraining, and constructing functions. Accordingly, representations can *complement* each other due to their different characteristics, which is, for instance, helpful for problem-solving processes. Moreover, one representation can *constrain* the interpretation of a second, which means, for example, that learners' familiarity with one representation can help interpreting a less familiar one. In this way, graphic representations can support the understanding of symbolic representations by bridging everyday thinking and formal thinking (Arcavi, 2003). Thus, combining symbolic and graphic representations can *construct* deeper understanding.

Ainsworth (2006) described different processes of how deeper understanding may be constructed with multiple representations: One process is *abstraction*, where learners construct references across representations to expose the underlying, more abstract structure of a mathematical concept (p. 189). Abstraction and concreteness fading are especially pronounced in traditional and current East Asian mathematics education: East Asian scholars have emphasized the aim of a transition from concrete to abstract representations (e.g., Ding & Li, 2014; Ding et al., 2022; Leong et al., 2015) and current East Asian textbooks focus on developing abstract and symbolic thinking (Ding et al., 2022; Yang et al., 2022). According to Ainsworth (2006), another process is *relational understanding*, identifying connections between representations, which can be the basis for abstraction, but also an end in itself. Especially Western mathematics education emphasizes this process as an end to itself: Many Western scholars have highlighted making connections between different representations of the same mathematical concept as an indicator of conceptual understanding and central for students' sense-making of mathematics (e.g., Duval, 2006; Noss et al., 1997). For instance, Noss et al. (1997) argued that for learning mathematics, the emphasis should be on connections between concrete and abstract representations rather than on the replacement of one by another. Moreover, making such connections was emphasized as a key process of mathematical work and especially relevant for mathematical modelling (e.g., Boaler, 2002; Chamberlin et al., 2022). In line with a focus on the process of doing mathematics (Leung, 2001), national standards in Western countries emphasize making connections between representations as an important overarching learning goal in mathematics (e.g., KMK, 2003; NCTM, 2000), which is typically not the case in East Asian countries such as Taiwan (National Academy for Educational Research, 2018).

2.2.2 The teacher's use of representations

Different features of *high-quality use of representations* in mathematics instruction can be derived from the functions of representations for learning mathematics: It is crucial that representations selected by the teacher are appropriate not only in terms of the mathematical content but also concerning students' understanding, the situation, and the learning goal (e.g., Hsieh, 2013; Mitchell et al., 2014). For example, one can argue a graphic representation of a balanced scale is not always an appropriate representation of a linear equation as negative numbers cannot be represented as weights on a scale. However, this representation can show the key features of equivalent equations as an equilibrium and its maintenance; therefore, it is internationally accepted as an appropriate representation for teaching how to transpose equations (e.g., Otten et al., 2019).

Traditionally, in East Asian countries like Taiwan it is more common to use mainly symbolic representations (e.g., Yang et al., 2022). However, since the 1980s, using more concrete representations has been emphasized due to Western influences – especially as a feasible start to foster non-elite students' understanding (e.g., Hsieh et al., 2018; Leong et al., 2015; Ministry of Education, 2000). Accordingly, teachers are expected to use graphic representations to introduce algebraic concepts, with standard textbook representations including scales or geometric shapes.

Nevertheless, the expectation for teachers to make explicit connections between graphic and symbolic representations to foster students' conceptual understanding appears to be advocated particularly by Western scholars (e.g., Dreher & Kuntze, 2015; Duval, 2006; Noss et al., 1997). This advocacy is supported not only theoretically but also empirically: Experimental studies have demonstrated that while the use of graphic representations alone may not significantly benefit learners' conceptual understanding, explicit encouragement to establish connections between graphic and symbolic representations is crucial (e.g., Rau & Matthews, 2017; Renkl et al., 2013). Other research designs gained further evidence: Suh and Moyer-Packenham (2007) showed in classroom projects how students' relational thinking and ability to translate between representations in algebra could be fostered by supporting them to make connections between graphic and symbolic representations. Zwetschler and Prediger (2013) found in a case study that facilitating conceptual understanding of the equivalence of algebraic terms by means of graphic representations of geometric shapes requires making explicit connections between symbolic and graphic representations and that learners often needed to be supported in making such connections.

Empirical studies examining teachers' use of representations in different cultural contexts also indicate systematic differences between Western and East Asian countries: Findings of the TALIS video study reveal that German teachers tend to encourage student engagement with graphic representations and use materials that make explicit connections among representations more often than teachers from other countries, such as China (OECD, 2020). Moreover, based on data from the Learners' Perspective Study, Huang and Cai (2011) compared a Chinese and a U.S. teacher's use of representations in sequences of lessons on linear relations. They found that in the U.S. classroom, multiple representations were constructed simultaneously and connections were emphasized, whereas in the Chinese classroom, selected representations were constructed hierarchically and symbolic as well as graphic representations (graphs of functions) were emphasized. Comparing these results with findings from previous studies of U.S. and Chinese teachers' use of representations for other mathematical concepts such as ratio, where Chinese teachers clearly preferred symbolic representations, they concluded that it appears to depend on the content whether Chinese teachers emphasize connections between graphic and symbolic representations. When graphic representations serve as an independent learning goal (e.g., graphs of functions), such connections appear to be more important than when graphic representations are merely an instructional aid in making sense of mathematics and facilitating conceptual understanding. (Huang & Cai, 2011).

In the light of these findings, questions arise regarding the transferability of Western research advocacy regarding explicit connections between representations to East Asian countries like Taiwan, particularly concerning graphic representations that are not considered an independent learning goal.

Furthermore, even for Western countries, there remains ambiguity regarding whether advocating for explicit connections between representations constitutes a norm of high-quality use of representations in the mathematics education community. Some Western scholars focus solely on selecting appropriate representations for teaching when considering teachers' use of representations (e.g., Kunter et al., 2013). And even if using multiple representations is emphasized, making connections between representations is not always a decisive criterium: For instance, Schoenfeld (2018) pointed out that earning high scores on his framework of instructional quality demands that multiple representations be connected, whereas the Mathematical Quality of Instruction (MQI) framework (Hill et al., 2008) gives high scores if different representation are discussed in a lesson.

3 Research questions

This study aims to compare the expectations of German and Taiwanese researchers regarding teachers' use of representations in the mathematics classroom, focusing on whether the Western advocacy of making explicit connections between representations is reflected in their instructional quality norms.

The following questions guide this investigation:

RQ1. Do German/Taiwanese mathematics education researchers' evaluations of classroom situations suggest that making explicit connections between symbolic and graphic representations can be considered a norm of high-quality use of representations in their mathematics education community?

RQ2. Apart from making connections between representations, what expectations regarding the teachers' use of representations emerge from the researchers' evaluations of the classroom situations?

RQ3. How do the expectations regarding the teachers' use of representations, including making connections and other expectations, differ between the two cultural contexts across the situations?

4 Methods

4.1 Wider project context

This study is part of a research project investigating the culture-specificity of norms of instructional quality by contrasting Germany and Taiwan.

As norms of instructional quality are usually tacit and bound to situations, investigating them requires a situated approach (Herbst & Chazan, 2011). Therefore, Herbst and Chazan (2011) proposed a method based on the notion of a breaching experiment (Mehan & Wood, 1975) to make norms of instructional quality accessible for mathematics education research. The main idea is that the reaction of a person confronted with a breach of a potential norm in a situation indicates whether this person expected the norm to hold: If the person brings up the issue and demonstrates criticism, the person expected the norm to hold. Herbst and Chazan (2011) suggested using vignettes of classroom situations in which a potential norm of instructional quality is breached. This approach is used in our project to elicit researchers' norms of instructional quality. The vignettes were designed by our German-Taiwanese research team with a methodology of concurrent development sensitive to different cultural contexts, which was described in detail by Dreher et al. (2021). In the following, we briefly sum up the process:

(1) Aspects of instructional quality that are central in both cultures, but also prone to potentially different norms were chosen: responding to students' thinking (Dreher et al., 2021), use of representations (this study), and use of tasks (Paul et al., 2024). We chose the topic of functions and equations as it is central in the secondary curriculum and different kinds of graphic representations are used to teach it in Germany and Taiwan.

(2) Each national team designed text-vignettes of classroom situations with a *breach of an anticipated norm* regarding a specific aspect of instructional quality from their perspective (3 per aspect and country). External feedback by practitioners in the country ensured ecological validity.
(3) The vignettes were translated into English. Each team checked whether the vignettes from the other team could represent practice in their country and whether adequate terms existed in their language. If necessary, vignettes were revised in an iterative process to represent practice and pedagogical lexicons (Mesiti et al., 2021) in both cultures.
(4) The national teams translated the revised foreign vignettes into their language. We subjected the German and Chinese versions of all vignettes to a check of linguistic equivalence (ITC, 2017: TD-2) by a person with a German–Taiwanese, mathematics, and educational background.

For the use of representations, this process yielded three vignettes from each national team. While the Taiwanese vignettes focused mainly on situations in which, from their perspective, a representation was used in a way that was not appropriate in terms of content, the German vignettes focused on situations in which, from their perspective, graphic and symbolic representations were not connected sufficiently (*breach of the anticipated norm*). These different focuses may already indicate culture-specific perspectives. Lindmeier et al. (2024) present findings regarding the vignettes authored in Taiwan. According to the research interest of this contribution, we focus on the vignettes by the German team to illustrate the validity problems of the typical scenario where norms of Western researchers are used in cross-cultural research.

4.2 The vignettes

4.2.1 Vignette 1

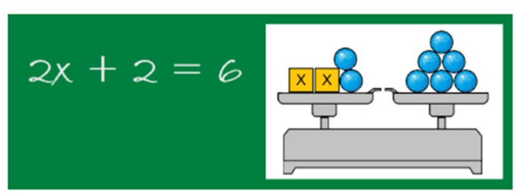
The situation represented in Fig. 1 focuses on the introduction of solving algebraic equations. The teacher uses an image of a balanced scale with objects (of known and unknown weights) on both sides to visualize the equation. Using this kind of graphic representation to introduce how to solve equations is common in Germany and Taiwan. The function of this graphic representation is to facilitate conceptual understanding of the equivalence of equations and how to transpose equations, but it is not an autonomous learning goal (Otten et al., 2019). From the perspective of the German team, to facilitate conceptual understanding, it is essential that the connections between the (mental) manipulations on the scale and the transformations of the equations are made explicit. In the vignette, they feel that this anticipated norm is breached: For instance, the teacher just claims that S2 has "divided by 2 on both sides" without explicitly connecting this to its meaning in terms of manipulating the scales. Another indicator for this suboptimal connection is that the teacher writes down the new equations in the process of transformations without displaying how the objects on the scales change.

4.2.2 Vignette 2

The situation represented in Fig. 2 focuses on the activation of prior knowledge regarding the first binominal formula with the aim of introducing the technique of "completing

Fig. 1 Vignette 1

Teacher T introduces *solving algebraic equations*. As an introductory example T writes the equation $2x + 2 = 6$ on the board and puts a picture with scales that illustrates this equation next to it on the blackboard:

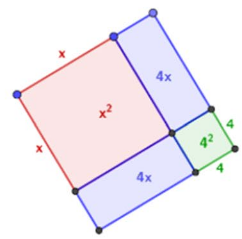


T: Here you can see an equation, two terms connected with an equal sign. We can imagine that this equation is like scales that are in balance. So the weight on the left and on the right is equally heavy. The balls have all the same weight of 1 unit weight each. This means there are 6 unit weights on the right side. And we have two boxes of the same unknown weight x. Finding out how heavy x must be for the scales to be in balance can help us to solve the algebraic equation. So what can we do to find out the weight of x?
S1: We could take away two balls. I mean, um, on both sides of the scales? This should keep the scales in balance and then we will have only boxes on the left side.
T: Good idea. So we take minus 2 on each side. [Writes $2x = 4$ on the blackboard.] And now? S2?
S2: Ok, so x must be 2.
T: Yes, good! So you divided by 2 on both sides. [Writes $x = 2$ on the blackboard.] We have solved the equation! So x is 2. This means a box weighs as much as two balls, hence two unit weights.

Fig. 2 Vignette 2

The teacher T plans the introduction of the technique *completing the squares* for solving *quadratic equations.* In order to activate necessary prior knowledge, T starts with a recap regarding the rationale behind the binomial formula.

T: Do you remember this kind of visualization that we used last year? What was this about? [T displays the visualization for everyone.]



S1: I think it was about the binomial formula. [Students nod.]

T: Yes, this is correct! The binomial formula. You know, they are all about restructuring quadratic terms. What algebraic term could we write to match this visualization? [5 sec silence]

S2: $x^2 + 8x + 4^2$.

T: Very good! So how can we restructure this term?

S3: This is the same as $(x + 4)^2$.

T: Great! So we will need this a lot in the following lessons and it may be helpful to use the visualization again.

the squares" for solving quadratic equations. The teacher displays a graphic representation of a square used in this class previously for introducing the binomial formula. In this context, the graphic representation is a means for facilitating conceptual understanding of the binomial formula by showing that both terms of the binomial formula are equivalent since they can describe the area of the same square (Zwetschler & Prediger, 2013).

From the perspective of the German team, for using this potential, it is essential that the connections between the symbolic terms and how they are represented in the graphic representation are made explicit. In the vignette, they feel that this anticipated norm is breached: It is not explained how the term S2 proposes can be seen in the graphic representation, and, even more importantly, it is not clear how S3 got the second term (symbolic transformation or from the graphic representation) and how this can be seen in the graphic representation.

4.2.3 Vignette 3

The situation represented in Fig. 3 focuses on introducing stretched and compressed parabolas. The teacher uses a dynamic visualization to show how changing the factor *a* in the functional equation affects the shape of the function's graph. Graphs as graphic representations of functions are not only a means to facilitate conceptual understanding of functions but are also an autonomous learning goal (Huang & Cai, 2011). In this classroom situation, the students are supposed to learn how the graphs of quadratic functions look like depending on the factor *a*. From the perspective of the German team, for the students to understand the reason for the effect of varying *a* on the shape of the graph, it is essential to make explicit connections between the symbolic and the graphic representation pointwise (and not only regarding the graph as a whole). In the vignette, they feel that this anticipated norm is breached since the students merely observed on a phenomenon level what happens to the graph when *a* is varied. The teacher explains what happens in the symbolic representation, but explicit connections between the symbolic and the graphic representation regarding specific values of *x* are missing.

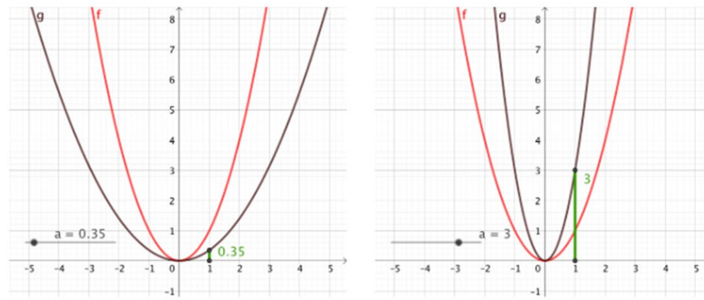
4.3 Participants and data collection

These vignettes were presented to researchers in Germany and Taiwan to find answers to the research questions. For the online survey, we recruited professors active in mathematics education research and in preparing future mathematics teachers. As we aimed for a sample of 15 professors in each country and assumed a participation rate of 50%, in Germany, a random sample of 30 professors out of the full list of persons meeting these criteria was contacted. In Taiwan, these criteria yielded a list of only 32 professors; thus, they were all contacted. In total, a sample of $n_1 = 19$ Taiwanese professors from 10 universities and a sample of $n_2 = 12$ German professors from 10 universities worked on the vignettes (completion rates were TW 59%, GER 40%). Besides educating mathematics pre-service teachers, most professors also had experience as schoolteachers (TW 14, GER 12) and in giving teacher trainings (TW 17, GER 11).

Regarding each vignette, they were given the following open-ended prompt: "Please evaluate the teacher's use of representations in this situation and give reasons for your

Fig. 3 Vignette 3

After the introduction of the graph of the function $f(x) = x^2$, today's lesson is about the introduction of *stretched and compressed parabolas*. The teacher T uses a dynamic visualization. The factor a in $g(x) = ax^2$ can be varied dynamically in the visualization. [T displays the visualization for everyone.]



T: Okay, the red graph belongs to the function $f(x) = x^2$, as we know already. And now we have a factor a, which we can adjust here. [T moves the regulator for a slowly back and forth to vary g.] What do you observe?

S1: If you make a bigger, the parabola gets narrower.

S2: And if you make a smaller, the parabola gets broader. Umm, I mean, if a is between 0 and 1, the black parabola is broader than the red parabola.

T: These are very good observations! The black parabola is the graph of the function $g(x)= ax^2$. [T writes on the board.] Can you describe in your own words what this factor a does? [5 sec silence]

S3: Umm well, everything gets multiplied by a. The x^2 is multiplied by a.

T: Exactly. And we remember: Multiplication with a number > 1 makes bigger, and multiplication with a number between 0 and 1 makes smaller. So this means if we take an a bigger than 1, then the values of g will be bigger than the values of f. And if we take an a between 0 and 1, the values of g will be smaller than the values of f. This is exactly what we observed in the visualization.

answer." The vignettes were presented in randomized order and could be worked on at the participants' own pace. The survey was conducted in the participants' mother tongue.

4.4 Coding processes

To enable the whole team to code all answers and directly compare them across cultures, all answers were translated into English and the external trilingual person compared the translations from Chinese to English with those from German to English. Both language versions were used in parallel during the coding processes.

A deductive coding process was applied to identify whether a participant's evaluation of the teacher's use of representations indicated that this participant expected the anticipated norm to hold (RQ1). This coding process consisted of two steps and was applied independently by each team member to all participants' answers. The first step was a dichotomous coding of whether the participant saw some aspect of the use of representations critically. If so, in a second step, it was coded why the use of representations was criticized as an indicator of whether the participant brought up the issue corresponding to the anticipated norm (i.e., connections between the symbolic and the graphic representation were not explicit enough).

To figure out whether making explicit connections between symbolic and graphic representations can be considered a norm of high-quality use of representations in the German/Taiwanese mathematics education community, we checked regarding each vignette, how many of the researchers in each country noticed a breach of the anticipated norm (RQ1). For interpreting these results, it is important to consider that even if a norm exists in a mathematics education community, it cannot be expected that all of the researchers' answers reveal that they noticed the corresponding breach of this norm for several reasons. Firstly, there are always individuals not agreeing with commonly accepted norms in their cultural context. Secondly, not only had the researchers to notice that no explicit connections between the representations were made, but they also had to express this clearly in their written answers, despite other salient features in the classroom situations. Therefore, we determined that if at least 50% of the researchers' evaluations from one country show that they actively recognized the breach of the anticipated norm, then this could be considered indicative of the existence of this norm in the corresponding culture (Dreher et al., 2021).

To get further insight into the participants' expectations regarding the teacher's use of representations in the vignettes (RQ2), we also applied inductive coding: By means of structuring qualitative content analysis (Kuckartz & Rädiker, 2023), we extracted for each vignette other reasons why participants criticized the use of representations and aggregated similar reasons in categories. These categories, hence, correspond to further expectations regarding the use of representations in each vignette. It turned out that partly similar expectations were extracted for the different vignettes, so we decided to apply the emerging set of categories as additional codes in the second coding step across the vignettes.

The codings were first compared within the national teams, and discrepancies were resolved through discussion. Then, the national coding results were compared. For both coding steps and all three vignettes, the cross-cultural inter-rater-reliability was good (Cohen's kappa $\geq .77$). In the case of discrepancies, a consensus was reached through discussion, where the interpretation of the team from the culture of the participant was usually given priority.

Finally, by comparing the results (RQ1 and RQ2) regarding the researchers' evaluations across all vignettes, we compare how the expectations regarding the teachers' use representations differ between the two cultural contexts (RQ3).

5 Results

In the following, we will first report the coding results for each vignette, and in the end, we provide a synopsis across all three vignettes to answer the research questions.

5.1 Vignette 1

Regarding the classroom situation of vignette 1, 75% of the 12 German researchers' answers and 63% of the 19 Taiwanese researchers' answers criticized aspects of the teacher's use of representations.

The second coding step revealed that 58% of the German and 21% of the Taiwanese researchers saw a breach of the anticipated norm (insufficient connection between symbolic and graphic representation). Figure 4 shows an example of a corresponding answer of a German researcher (left: English translation, right: original answer).

This researcher criticizes the teacher's use of representations and concludes that this probably means that the students "do not benefit enough from the use of the scale model". One of the reasons for the negative evaluation is that the connections between the graphic representation ("scale model") and the symbolic representation are not explained sufficiently (regarding the subtraction, the division, and the solution). This indicates that this researcher recognized a breach of the anticipated norm.

Three inductively extracted categories (IC) reflecting further expectations regarding the teachers' use of representations occurred for this vignette. Another point of criticism mentioned by the researcher in Fig. 4 is that the translations between the representations are done by the teacher and not by the students. A similar point was made by two further German and one Taiwanese researcher (IC1: *The students should be more active in making connections between the representation*s). The remaining criticisms referred to the selection of representations and are illustrated by the sample answers in Fig. 5: Four German and one Taiwanese researchers criticized that the representation of the scale is static (IC3: *Representation should be dynamic*). About a quarter of the researchers in each country pointed out weaknesses of the graphic representation like the restriction that this representation does not work for negative numbers or that maybe students are not familiar with such scales (IC2: *weaknesses of the graphic representation*).

5.2 Vignette 2

Regarding the classroom situation of vignette 2, 67% of the German researchers' answers and 58% of the Taiwanese researchers' answers criticized aspects of the teacher's use of representations.

The second coding step revealed that 50% of the German and 21% of the Taiwanese researchers saw a breach of the anticipated norm (insufficient connection between symbolic and graphic representation). Figure 6 shows an example of a corresponding answer of a German researcher.

In this answer, the researcher criticizes that the potential of the area representation to facilitate conceptual understanding regarding the equivalence of the terms is not used. Furthermore, it is explained in detail that the connections between the graphic and the symbolic representation are not made explicit enough.

Two of the inductively generated categories for vignette 1 also occurred regarding this vignette: One researcher from each country mentioned that *the students should be more active in making connections between the*

Fig. 4 Sample answer of a German researcher recognizing a breach of the anticipated norm. The answer also illustrates IC1

[...] The translation of the subtraction from the scale model to the algebraic term is done by T and is not explained. The solution offered by S2 is explained symbolically by T (divided by 2), without explaining how division is understood in the scale model. Also, the solution x is 2, is translated to the scale model by T. The essential steps are not performed by the S [plural form], but by the teacher. It can be assumed that the S do not benefit enough from the use of the scale model.	[...] Die Übersetzung der Subtraktion vom Waagemodell in den algebraischen Term wird von der L. vollzogen und nicht begründet. Die angebotene Lösung von S2 wird von der L symbolisch begründet (durch 2 geteilt), ohne zu begründen, wie Division im Waagemeodell [sic] zu verstehen ist. Auch die Lösung x ist 2, wird von der L ins Waagemodell übersetzt. Die wesentlichen Schritte leisten nicht die Sch, sondern die Lehrkraft. Es ist zu vermuten, dass die Sch zu wenig vom Gebrauch des Waagemodells profitieren.

Fig. 5 Sample answers of a Taiwanese and a German researcher regarding vignette 1 criticizing the selection of representations (IC2 and IC3)

A very natural initial-level representation. Currently, the restriction that x must be a positive quantity is not considered; moreover, the students do not doubt that the weight of each blue ball is equal.	很自然的初等表徵。此時並不深思 x 必為正量的限制，而且學生也不懷疑每個藍色球的重量相等。
[...] The representation of the scales is static, therefore the steps and ideas must be carried out without visible changes (imbalance purely hypothetical). Finally: It is only assumed that the scale represents the "equilibrium image", but it is not obvious why it works like this, perhaps students do not even know such scales with two trays from their everyday life.	[...] Die Darstellung der Waage ist statisch, daher müssen die Schritte und Vorstellungen ohne sichtbare Veränderungen durchgeführt werden (Ungleichgewicht rein hypothetisch). Schließlich: Von der Waage wird ausschließlich angenommen, dass sie das "Gleichgewichtsbild" darstellt, es ist aber nicht ersichtlich, warum sie so funktioniert, mglw. kennen SuS nicht einmal aus ihrem Alltag solche Waagen mit zwei Schalen.

Fig. 6 Sample answer of a German researcher regarding vignette 2 recognizing a breach of the anticipated norm

The teacher uses an area model as a representation of an algebraic expression. The S1-S3 recognize two terms and state the equality of the terms. The teacher takes this as a manifestation of understanding. However, the connections between term and image are not explicitly made (e.g., for other students), (a) where and how the two terms are found in the image, (b) that and why they represent sums of areas and (c) WHY they are equal. The function of area representations for justifying the equivalence of terms is not used in the sequence. [...]	Die Lehrkraft nutzt ein Flächenbild als Repräsentation eines algebraischen Ausdrucks. Die S1-S3 erkennen zwei Terme wieder und stellen die Gleichheit der Terme fest. Die Lehrkraft nimmt das als Ausdruck des Verständnisses. Die Beziehungen zwischen Term und Bild werden aber nicht explizit hergestellt (zB für andere SuS), (a) wo und wie die beiden Terme sich im Bild wiederfinden, (b) dass und warum sie Summen von Flächeninhalten darstellen und (c) WARUM sie gleich sind. Die Funktion von Flächendarstellungen zur Begründung der Gleichwertigkeit von Termen wird in der Sequenz nicht genutzt. [...]

Fig. 7 Two sample answers of Taiwanese researchers regarding vignette 2 illustrating IC2 and IC4

The use of representations for activating prior experience and knowledge was quite successful. Nonetheless, why was the square oblique? This could make the representation difficult to read and comprehend.	喚起先備經驗與知識，這部分的表徵使用相當成功。只是，為何正方形以傾斜方式展示？可能增加識讀上的困難。
Instead of presenting the graphical representation of an intact large square in the beginning, the teacher should have prepared picture cards that corresponded to the algebraic expression. That is, to prepare a square figure of an x square unit, a rectangular figure of an 8x unit (or cut it into two 4x rectangles first according to students' levels), and a square figure of a 4 square unit. Then, let the students try to make a large square and observe the side length of the large square, as well as compare the square assembled with the original algebraic expression. Teaching activities of this type could provide a better link with the method of completing the square to be introduced later.	該教師可以先不要一開始就呈現完整的大正方形圖形表徵。應先準備對應代數式的圖形卡，也就是準備一張 x 平方的正方形圖形，一張 8x 的長方形圖形(考量班級學生程度或先裁成兩張 4x 長方形圖形)和一張 4 平方的正方形圖形，讓學生嘗試拼成大的正方形，再觀察所拼出來的大正方形邊長，所形成的面積和原來的代數式相比較，這樣的教學活動較能與之後要介紹的配方法相連結。

representations (IC1). Four Taiwanese and two German researchers pointed out *weaknesses of the graphic representation* (IC2). Figure 7 shows two examples of such answers. The first Taiwanese researcher criticizes that the square is not positioned in the standard way since this could cause unnecessary difficulties in interpreting the representation. The second Taiwanese researcher also mentions that the graphic representation is not optimal. By suggesting an alternative representation, the researcher also argues that this graphic representation is not very suitable for introducing the method of completing the square. Such criticism that *the use of representations does not clearly scaffold the objective* of the teaching sequence (IC4) was mentioned by two German and three Taiwanese researchers.

5.3 Vignette 3

Regarding the classroom situation of vignette 3, 67% of the German and 79% of the Taiwanese researchers criticized the teacher's use of representations.

For this vignette, the second coding step revealed that 58% of the German, as well as 58% of the Taiwanese, researchers saw a breach of the anticipated norm (insufficient connection between symbolic and graphic representation). Figure 8 shows examples of corresponding answers

Fig. 8 Sample answers of a Taiwanese and a German researcher regarding vignette 3 recognizing a breach of the anticipated norm. The lower answer also illustrates IC1

The teacher used dynamic graphs to present the graphs of functions for the students to observe, which was helpful for students to perceive the classification of the coefficients of quadratic terms and changes in graphic appearance, as well as the relationship between the changes in the coefficients of quadratic terms and the changes in the function value. However, clear representation transformations would be required to show the relationships among the changes in function values, point coordinates, and graphs.	教師使用動態圖形展示函數圖形給學生觀察，有助於學生察覺到二次係數與圖形樣貌變化的分類，以及二次係數與函數值的變化關係，但函數值、點坐標及圖形三者變化的關係，還需要清楚的表徵轉換
In my opinion, the teacher does not sufficiently use the possibilities of the used representations and the changes between them in a didactically meaningful way. Although the changes of the graphs depending on the parameter a are addressed at the beginning, as soon as he or she has written the function term on the blackboard, the graphical representation does not play a role anymore, as only the effect of the multiplication of x^2 with different factors is discussed. In order to understand the relationships between the change of the graphs and the effect of the factor on the function values, it is essential to explicitly investigate several parabola points of the normal parabola with the corresponding points of the changed parabola. Only in this way it is possible to establish the notion of "compressing" a parabola as "pressing down the y-coordinate from above at a specific x-value" or of "stretching" as "pulling up the y-coordinate at a specific x-value". This idea should then necessarily be consolidated by at least one other example (each with other values for a) before relating the connections between this procedure and the effect on the purely symbolic level. The latter should ideally be done by the students themselves.	Ich finde, dass die Lehrkraft die Möglichkeiten der hier genutzten Repräsentationen und der Übergänge zwischen ihnen nicht in ausreichendem Maße didaktisch sinnvoll nutzt. Zwar werden die Veränderungen der Graphen in Abhängigkeit des Parameters a zu Beginn thematisiert, sobald er oder sie aber den Zuordnungsterm an die Tafel geschrieben hat, spielt die graphische Repräsentation keine Rolle mehr, denn es wird nur über die Auswirkung der Multiplikation von x^2 mit verschiedenen Faktoren gesprochen. Um die Zusammenhänge zwischen der Änderung der Graphen und der Auswirkung des Faktors auf die Funktionswerte zu verstehen, müssten unbedingt mehrere Parabelpunkte der Normalparabel mit den entsprechenden Punkten der geänderten Parabel explizit untersucht werden. Nur so kann die Vorstellung des "Stauchens" einer Parabel als "Runterdrücken der y-Koordinate von oben an einem bestimmten x-Wert" oder das "Strecken" als "Hochziehen der y-Koordinate an einem bestimmen x-Wert" aufgebaut werden. Diese Vorstellung müsste dann noch über mindestens ein anderes Bespiel (jeweils andere Werte für a) gefestigt werden und erst dann die Zusammenhänge dieses Vorgehens mit der Auswirkung auf der rein symbolischen Ebene in Zusammenhang gebracht werden. Letzteres sollte idealerweise durch die Schülerinnen und Schüler selbst erfolgen.

of a Taiwanese and a German researcher. Both criticize that the connections between the representations were not made explicit enough, particularly regarding specific values of x.

Two of the inductively generated categories were also found for vignette 3. Criticisms pointing out *weaknesses of the graphic representation* (IC2, 1 German, 4 Taiwanese researchers) referred to the use of colors or labeling of the graph and were always combined with recognizing a breach of the anticipated norm. Moreover, five German researchers pointed out that *the students should be more active in making connections between the representations* (IC1). The last sentence of the German researcher's answer in Fig. 8 is an example of mild criticism in this regard. No further categories were found inductively for this vignette.

5.4 Synopsis of findings

The synopsis of our findings across the vignettes in Table 1 shows for RQ1, that regarding each vignette, at least half of the German researchers recognized a breach of the norm anticipated by the German team concerning the connections between symbolic and graphic representation. This indicates that this may be considered indeed a norm of high-quality use of representation in Germany. In Taiwan, only regarding the third vignette, where the graphic representation is considered an autonomous learning goal, did the majority of the researchers recognize the anticipated breach of a norm. Hence, explicitly connecting symbolic and graphic representations seems not to be expected by the Taiwanese researchers in cases where graphic representations only serve as a means for constructing meaning. However, expectations regarding the connection of representations seem to be different in the case where the graphic representation is considered an autonomous learning goal.

To answer RQ2, we consider the inductively generated categories IC1-IC4 reflecting further expectations regarding the teacher's use of representations in the classroom situations. The expectations that students should be more active in making connections between representations (IC1) and that weaknesses of the graphic representation should be addressed (IC2) were mentioned regarding all three situations. Regarding the first situation, it was also expected that the graphic representation should be dynamic (IC3), and regarding the second situation, it was additionally expected that the use of representations should better scaffold the objective of the teaching sequence (IC4). Notably, in all situations, further expectations were expressed less frequently than the expectation of more explicit connections between symbolic and graphic representation (breach of anticipated norm).

The comparison between German and Taiwanese researchers' evaluations of the teachers' use of representations across the situations (RQ3) regarding the inductively generated categories can only be made cautiously due to the small numbers of answers for the inductively generated categories. However, it may be observed that IC2 and IC4 were mentioned by researchers from both countries to a similar degree, whereas IC1 and IC3 were mentioned mainly by German researchers.

6 Discussion

In this study, we examined whether perspectives on high-quality use of representations in the mathematics classroom vary across cultures. Specifically, we investigated expectations of German and Taiwanese researchers regarding teachers' use of representations, focusing on whether the Western

Table 1 Percentage of researchers that mentioned specific reasons for criticizing the use of representations regarding each vignette. Bold: coded for at least 50% of the Taiwan/German sample

	Vignette 1 (scales and equations)		Vignette 2 (binomial as square)		Vignette 3 (stretched parabola)	
	Germany	Taiwan	Germany	Taiwan	Germany	Taiwan
Insufficient connection between symbolic and graphic representation (breach of anticipated norm)	**58% (7/12)**	21% (4/19)	**50% (6/12)**	21% (4/19)	**58% (7/12)**	**58% (11/19)**
(IC1) Students should be more active in making connections	25% (3/12)	5% (1/19)	8% (1/12)	5% (1/19)	42% (5/12)	0% (0/19)
(IC2) Weaknesses of the graphic representation	25% (3/12)	26% (5/19)	17% (2/12)	21% (4/19)	8% (1/12)	16% (3/19)
(IC3) Representation should be dynamic	33% (4/12)	5% (1/19)				
(IC4) Lack of scaffolding the objective			17% (2/12)	16% (3/19)		
No criticism	25% (3/12)	37% (7/19)	33% (4/12)	42% (8/19)	33% (4/12)	21% (4/19)

advocacy for making explicit connections between representations is reflected in their instructional quality norms.

The results may serve three purposes: Firstly, they give insights into the differences in expectations regarding teachers' use of representations across mathematics education communities in different cultural contexts. Secondly, they prompt a deeper reflection on reasons for making explicit connections between symbolic and graphic representations, which may be linked to culturally shaped learning goals in mathematics education. Lastly, they highlight the challenge of applying Western perspectives on instructional quality to other cultural contexts, potentially leading to validity issues.

Before the results are discussed more detailed, we want to acknowledge some limitations of the study, urging caution in interpreting the evidence: While different expectations regarding teachers' use of representations were identified through an inductive approach, the findings may not provide a comprehensive picture as the study focused on the Western research advocacy of making connections between representations. Additionally, despite our sample strategies, the sample sizes are unequal, and the German sample may be considered rather small, making it partly difficult to judge what systematic differences are, especially regarding the inductively generated categories. Moreover, it could be argued that Taiwanese researchers might be reluctant to criticize due to their cultural background. However, there were only one Taiwanese and two German researchers who did not criticize the teacher's use of representations in any of the three vignettes. Regarding RQ1, we determined that making explicit connections between symbolic and graphic representations may indeed be considered a norm of high-quality use of representation in Germany. However, it is noteworthy that many German researchers also emphasized other aspects in their evaluations, indicating varying emphases among Western scholars as described above.

Nevertheless, the findings reveal both similarities and differences in expectations between German and Taiwanese researchers, which can be interpreted in the light of theoretical considerations and existing research.

Researchers from both countries articulated several expectations regarding teachers' use of representations that can be found in the mathematics education literature, including considerations for the selection of representations based on mathematical content, students' understanding, and the learning goal. As expected, differences between the German and Taiwanese researchers' evaluations occurred particularly regarding the expectation of making explicit connections between representations: While in each situation, at least half of the German researchers expected the teacher to make explicit connections between representations, only in one situation, the majority of the Taiwanese researchers expected this. One might argue that maybe such explicit connections are not considered relevant, as Taiwanese students can be expected to make these connections on their own based on their higher mathematical competence compared to their German counterparts (e.g., OECD, 2023). However, the fact that regarding the third situation, the majority of the Taiwanese researchers expected the same explicit connections as their German counterparts, indicates that the explanation is not as simple. Instead, in Taiwan, the perceived relevance of making explicit connections between symbolic and graphic representations appears to depend on the content and the role of the graphic representation in the classroom situation. Most of the Taiwanese researchers expected the teacher to make explicit connections between symbolic and graphic representations of a quadratic function, where the graph of the function is not a representation that is supposed to get replaced by a more abstract representation (e.g., Ding & Li, 2014). Unlike their Taiwanese counterparts, at least half of the German researchers also expected the teachers to make explicit connections in situations when the graphic representations were mainly aiding conceptual understanding and supporting students' sensemaking of the represented algebraic concepts. This result complements the findings by Huang and Cai (2011) based on case studies comparing Chinese and U.S. teachers and leads us to reflect on culture-specific values that may be behind these different expectations regarding making connections between symbolic and graphic representations.

As outlined above, there are different functions of making connections between representations for students' mathematical understanding, which have been emphasized to different degrees by Western and East Asian scholars. While Western scholars have emphasized making connections between symbolic and graphic representations as a learning goal, since it is considered an important process of doing mathematics and for meaningful learning (e.g., Boaler, 2002; Duval, 2006), East Asian scholars have rather emphasized making connections between representations with the idea of concreteness fading and aiming at abstract representations as a learning goal (e.g., Ding et al., 2022; Leong et al., 2015).

These different learning goals regarding the use of multiple representation in the mathematics classroom are probably related to the different expectations regarding the teacher's use of representations: From an East Asian perspective, if the graphic representation is considered an instructional aid on the way to abstraction, explicit connections do not appear necessary when the symbolic representations can be used already. From a Western perspective, if the students are supposed to learn how to make connections between concrete and abstract representations, optimally they actively make such connections, which was mainly expected by German researchers in this study. Moreover, the teacher is expected to support them in doing so. One means for such support is using dynamic representations (e.g., Kaput, 1992), which was also mentioned by some researchers regarding the first situation.

Although we think that general cultural differences cannot predict researchers' expectations regarding the teacher's use of representation in a straightforward way, it may be argued that the different learning goals identified regarding the use of multiple representations are not simply the result of different instructional emphases in two countries but are rooted in the corresponding cultures. As reasoned above, culture-specific perspectives on general learning goals in (mathematics) education (Hofstede, 1986; Leung, 2001) appear to be reflected in these more specific learning goals: The focus on processes of doing mathematics and the constructivist conceptions of meaningful learning resonates with the dominant Western perspective, whereas the emphasis on mathematics as a product and symbolic fluency is rather in line with the dominant East Asian perspective.

The finding that the Western research advocacy on making explicit connections between representations does not fully apply to the Taiwanese context, means that we need to be aware that frameworks and measures for instructional quality of mathematics classrooms may only be applicable in a specific cultural context, also regarding aspects of instructional quality where perspectives seem to be largely shared. Furthermore, empirical studies showing the effectiveness of making explicit connections between representations for students' learning also need to be seen within the corresponding cultural context and should be reflected regarding the learning goals that were focused on.

Funding Open Access funding enabled and organized by Projekt DEAL. This study is part of the project TaiGer Noticing funded by the DFG – German Research Foundation (grant numbers DR 1098/1–1 and LI 2616/2–1) and the Ministry of Science and Technology (MOST, grant number 106–2511-S-003–027-MY3).

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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